Extremum seeking via continuation techniques for optimizing biogas production in the chemostat

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Abstract

We consider the chemostat model with the substrate concentration as the single measurement. We propose a control strategy that drives the system at a steady state maximizing the gas production without the knowledge of the specific growth rate. Our approach separates the extremum seeking problem from the feedback control problem such that each of the two subproblems can be solved with relatively simple algorithms. We are then free to choose any numerical optimization algorithm. We give a demonstration for two choices: one is based on slow-fast dynamics and numerical continuation, the other is a combination of golden-section and Newton iteration. The method copes with non-monotonic growth functions.
Key-words. adaptive control, self-optimizing control, parameter optimization, biotechnology.

1 Introduction

The control design of chemostat models have been extensively studied in the literature, with the objective to provide efficient and reliable control strategies for industrial applications, such as biotechnological or pharmaceutical processes [9, 10, 15]. A common task is to drive a continuous stirred tank bioreactor to a set-point that optimizes a objective function, for instance the gas production rate [11, 43]. In this work, we consider the chemostat model with a single strain [44]

$$
\begin{aligned}
\dot{s} &= -\mu(s)b + u(s_{\text{in}} - s) \\
\dot{b} &= \mu(s)b - ub
\end{aligned}
\tag{1}
$$

where the state variables s and b denote the substrate and biomass concentrations, respectively (in these equations the yield coefficient has be chosen equal to one without any loss of generality). The concentration of substrate in the feed is denoted by s_{in}, and the dilution rate $u \geq 0$ is the manipulated variable. The production rate to be maximized is given by

$$
r = \mu(s)b \ .
$$

In such bioprocesses, if often happens that the growth kinetics $\mu(\cdot)$ is unknown (or poorly known) and possibly evolves slowly with time or changes of environment (temperature, pH...). Consequently, the robustness of the control strategies with respect to uncertainties on $\mu(\cdot)$ is a crucial issue for real applications. Many works have been done for the on-line estimation of the growth function [21, 22, 14, 47, 48, 35, 7] and the robust stabilization of such processes about a given reference point [13, 24, 39, 37, 40], but there are comparatively

much less works that concern the maximization of an objective under model uncertainties [16]. Such issues are typically addressed by the design techniques of extremum seeking controls [50, 8, 28, 26, 3, 4, 45, 46]. Roughly speaking, these techniques consist in adding an excitation signal to the input $u = \bar{u} + a\sin(\omega t)$, of small amplitude a and high frequency ω, and capturing an estimation of the gradient of the objective function by filtering. Three time scales (excitation signal, process, filtering) then operate in the overall closed loop system. Such a technique has been developed first in [49] for unknown functions $\mu(\cdot)$ of Monod or Haldane type. Later on, an adaptive extremum-seeking scheme has been proposed to improve the performances of the transient response, for the Monod's case in [52] and the Haldane's one in [32]. In these approaches, the production rate r is assumed to be measured on-line, and explicit expressions of the unknown function $\mu(\cdot)$ are required. An alternative extremum-seeking scheme combined with a neural network has been proposed in [20] to get out of these requirements. Nevertheless, all the mentioned approaches require the on-line measurement of the whole state (s, b), excepted in [33] where b only is measured but the result is dedicated to the Monod's kinetics. In the present paper, we consider that the single on-line measurement

$$y = s \tag{2}$$

is available. One may consider coupling the former extremum-seeking controllers with state observers [17, 9, 19, 12, 38, 36, 34, 25], but there is here an intrinsic difficulty due to the fact that "good" observers, i.e. whose speed of convergence can be arbitrary tuned, require the exact expression of the dynamics, and that "fast" observers are often sensitive to measurement noise and input disturbances.

In the present work, we present a rather different approach that does not require any knowledge on the growth function neither the measurement of the objective function, and that is based on a continuation method [1]. We do not consider any excitation signal. Instead, we design a dynamical extension of the chemostat model (1) in a slow-fast or "singularly perturbed" form [29], such that the attracting critical manifold is exactly the graph of the unknown function $s \mapsto \mu(s)$. Our extremum seeking scheme is an extension of a former work [41, 42] that gives such a method for the reconstruction of the graph of $\mu(\cdot)$ with the single measurement of the substrate, without any a priori knowledge on μ (excepted to be a smooth function). In particular, this technique can cope with non-monotonic growth functions and allows to identify unstable states of the open-loop system. The paper is organized as follows. In Section 2 we first present our general methodology, and then show how to apply it on the chemostat model in Section 5. Section 6 is devoted to numerical demonstrations.

2   Assumptions in the general framework

We consider a single-input/single-output dynamical system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{3}$$

($x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$) and an objective function

$$z = \phi(y) \tag{4}$$

to be optimized at steady state, i.e., we are looking for an output feedback controller that steers the state to an operating point $(x^\star, u^\star)$ that fulfills

$$\begin{aligned} &f(x^\star, u^\star) = 0 \\ &\phi(h(x)) \text{ is locally max. w.r.t. } u \in \mathbb{R} \text{ at } x^\star \end{aligned} \tag{5}$$

where the functions f and ϕ are unknown or partially known.

We follow the statement of the general nonlinear problem of extremum seeking given in [3], that we adapt here to an output feedback framework.

Assumption A1 (Existence of stabilizing output feedback). There exists a smooth output feed-back $u(t) = \alpha(y(t), \bar{u}, \bar{y})$ parametrized by the parameter pair $(\bar{u}, \bar{y}) \in \mathcal{U} \times \mathcal{Y}$ (a pair of reference input and output), such that

$$\alpha(\bar{y}, \bar{u}, \bar{y}) = \bar{u} \ , \tag{6}$$

and the closed loop system

$$\dot{x} = f(x, \alpha(h(x), \bar{u}, \bar{y})) \tag{7}$$

admits a unique equilibrium $x_{eq}(\bar{u}, \bar{y})$, that is locally asymptotically stable for any $(\bar{u}, \bar{y}) \in \mathcal{U} \times \mathcal{Y}$.

For example, for the chemostat $(1), (2)$ the output feedback could be of the form $u(t) = \bar{u} + G_1(\bar{s} - s(t))$ with a sufficiently large G_1 [42].

Under Assumption A1, we consider the extremum-seeking problem for the closed-loop dynamics (7) as if the pair $(\bar{u}, \bar{y})$ was a new control, and look for pairs such that $h(x_{eq}(\bar{u}, \bar{y})) = \bar{y}$. For this purpose, we shall consider the output-input characteristics defined as follows.

Assumption A2 (Output-input characteristics). There exists a smooth function $\psi : \mathcal{Y} \mapsto \mathcal{U}$ such that

$$\bar{y} = h(x_{eq}(\bar{u}, \bar{y})) \iff \bar{u} = \psi(\bar{y}) \tag{8}$$

for any $\bar{y} \in \mathcal{Y}$.

This assumption means that for each parameter $\bar{y} \in \mathcal{Y}$, there exists an unique input $\bar{u} \in \mathcal{U}$ that is a zero of the function $u \mapsto h(x_{eq}(u, \bar{y})) - \bar{y}$. For instance, in the chemostat model, positive equilibriums have to fulfill $\alpha(s_{eq}, \bar{u}, \bar{s}) = \mu(s_{eq})$, and then having $s_{eq} = \bar{s}$, along with (6), amounts to write $\bar{u} = \mu(\bar{y})$.

Assumption A3 (Existence of local maximum). There exists $y^\star \in \mathcal{Y}$ such that the function $\phi(\cdot)$ has a local strict maximum at $y^\star$.

Finally, we shall assume that the unknown objective function $\phi(\cdot)$ possesses a known structure.

Assumption A4 (Structure of objective function). There exists a smooth function $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\phi(y) = \varphi(y, \psi(y)) \ , \forall y \in \mathcal{Y} \tag{9}$$

Assumptions A2 and A4 imply that the extremum seeking problem amouts to optimize a known function of y and u at steady state. Thus, the problem consists then in finding an output feedback strategy that drives the state of the system to $x^\star$ such that $h(x^\star) = y^\star$ using the knowledge of the feedback law α and the objective φ only. The following two sections present two approaches.

3   Continuous-time adaptation

The continuous-time approach first constructs a differential equation for $\bar{u}$ that achieves $\bar{u} = \psi(\bar{y})$ (and, thus, $\bar{y} = h(\bar{x})$) asymptotically, and then applies a gradient search (again through a differential equation) along the curve of $(\bar{u}, \bar{y})$ given implicitly by $\bar{y} = h(\bar{x}(\bar{u}, \bar{y}))$. Note that for feedback laws of the form $u(t) = \bar{u} + G[\bar{y} - y(t)]$ this implicit curve corresponds to the curve of open-loop equilibrium outputs.

3.1   Step 1: an adaptive scheme for $\bar{u}$

We look for a dynamics

$$\dot{\bar{u}} = \beta(y, \bar{u}, \bar{y}) \tag{10}$$

with $\beta(\bar{y}, \bar{u}, \bar{y}) = 0$, such that

$$\bar{E}(\bar{y}) = \left[\begin{array}{c} x_{eq}(\psi(\bar{y}), \bar{y}) \\ \psi(\bar{y}) \end{array} \right] \tag{11}$$

is a locally asymptotically stable equilibrium of the coupled dynamics

$$
\begin{aligned}
\dot{x} &= f(x, \alpha(h(x), (\bar{u}, \bar{y}))) \\
\dot{\bar{u}} &= \beta(h(x), \bar{u}, \bar{y})
\end{aligned}
\tag{12}
$$

Note that Assumptions A1 and A2 ensures that $E(\bar{y})$ is an equilibrium of (12) for any $\bar{y} \in \mathcal{Y}$, and that one has

$$
\lim_{t \to +\infty} y(t) = \bar{y} \; .
\tag{13}
$$

This is a classical adaptive output control problem, for which several techniques are available in the literature [5, 27]. We simply require the convergence

$$
\lim_{t \to +\infty} \bar{u}(t) = \psi(\bar{y})
\tag{14}
$$

to be uniform w.r.t. $\bar{y} \in \mathcal{Y}$.

3.2   Step 2: a continuation method for $\bar{y}$

Step 1 with Assumption A4 provides an approximation of the unknown objective function $\phi(\cdot)$ at a fixed $\bar{y}$:

$$
\phi(\bar{y}) \simeq \bar{\phi}(\bar{y}) = \varphi(\bar{y}, \bar{u}) \; .
\tag{15}
$$

The continuation consists in having $\bar{y}$ evolving slowly (step-wise or continuously) with a steepest descent based on an estimation of the gradient of the function ϕ. A continuous adaptation can be written as

$$
\dot{\bar{y}} = -\epsilon \left[\partial_1 \varphi(\bar{y}, \bar{u}) + \partial_2 \varphi(\bar{y}, \bar{u}) \hat{\psi}'(\bar{y}, \bar{u}) \right]
\tag{16}
$$

where $\epsilon > 0$ is small compared to the time scale of dynamics (12), and $\hat{\psi}'(\bar{y}, \bar{u})$ is an estimation of the derivative of the function ψ at $\bar{y}$, for which $\bar{u}$ is an estimation of $\psi(\bar{y})$. Several gradient estimations techniques are available in the literature [30, 31, 51, 18, 6]. In the scalar case, an estimation of the sign of the derivative of the function ϕ is enough to choose the steepest descent. We can use for instance a dynamics with delay:

$$
\begin{aligned}
\tfrac{1}{\epsilon} \dot{\bar{y}} &= \delta(\bar{y}, \bar{u}, \bar{y}(t - \tau), \bar{u}(t - \tau)) \\
&= \operatorname{sgn}\left[(\varphi(\bar{y}, \bar{u}) - \varphi(\bar{y}(t - \tau), u(t - \tau))(\bar{y} - \bar{y}(t - \tau)) \right] \; .
\end{aligned}
\tag{17}
$$

Then, the overall dynamics

$$
\begin{aligned}
\dot{x} &= f(x, \alpha(h(x), \bar{u}, \bar{y})) \\
\dot{\bar{u}} &= \beta(h(x), \bar{u}, \bar{y}) \\
\dot{\bar{y}} &= \epsilon \delta(\bar{y}, \bar{u}, \bar{y}(t - \tau), \bar{u}(t - \tau))
\end{aligned}
\tag{18}
$$

is slow-fast where $\bar{u} = \psi(\bar{y})$ is the (attracting) critical manifold.

Finally, the output feedback strategy that we consider takes the following form

$$
u = \alpha(y, \bar{u}, \bar{y}) \text{ with }
\begin{cases}
\dot{\bar{u}} = \beta(y, \bar{u}, \bar{y}) \\
\dot{\bar{y}} = \epsilon \delta(\bar{y}, \bar{u}, \bar{y}(t - \tau), \bar{u}(t - \tau))
\end{cases}
\tag{19}
$$

Let us underline that the output y is naturally filtered by the dynamics to obtain the estimation $y^\star$ as

$$
\hat{y}^\star = \lim_{t \to +\infty} \bar{y}(t)
\tag{20}
$$

that provides a robustness w.r.t. to measurement noise. Of course, the price to pay is to have a slow convergence due to small ϵ.

4

4 Extremum-seeking using numerical optimization algorithms

We compare the continuous-time adaptation with classical numerical optimization algorithms that are difficult to express as dynamical systems such as golden-section iteration (which is applicable for single-parameter problems) and Newton iterations. These algorithms do not require any knowledge of the underlying system apart from the ability to evaluate the asymptotic output $\lim_{t\to\infty} y(t)$ for any given admissible inputs $(\bar{u}, \bar{y})$. Once the stabilizing feedback law $u(t) = \alpha(y(t), \bar{u}, \bar{y})$ is implemented, the problem is reduced to a classical numerical optimization problem: $\phi(h(x_{eq}(\bar{u}, \bar{y}))) \to \max$. In fact, if the feedback law is linear of the type

$$u(t) = \alpha(y(t), \bar{u}, \bar{y}) = \bar{u} + G[\bar{y} - y(t)] \tag{21}$$

this is an optimization problem in the single parameter $\bar{v} = \bar{u} + G\bar{y}$:

$$\phi(h(x_{eq}(\bar{u} + G\bar{y}))) =: F(\bar{v}) \to \max \quad \text{w.r.t } \bar{v}. \tag{22}$$

We test a combination of two classical optimization algorithms. We start with an initial admissible interval $[\bar{v}_{\text{low}}, \bar{v}_{\text{up}}]$. Then we use golden section search to narrow the initial interval down to a given fraction of its original size and then apply a Newton iteration to solve $F'(\bar{v}) = 0$ in the remaining interval. Neither of these two methods requires knowledge about the structure of F, only the ability to evaluate F and its derivatives at any desired point (where derivatives can be approximated by finite differences). For the evaluation of F and its derivatives in a given point $\bar{v}_0$ we use a naive *act-and-wait* approach [23]: one sets the parameter $\bar{v} = \bar{v}_0$, such that the feedback control law is $u(t) = \bar{v}_0 - Gy(t)$, waits until the transients have settled (such that $x = \bar{x}_0 := x_{eq}(\bar{v}_0)$), and then reads off $y = h(\bar{x}_0)$ and evaluates $\phi(y) = \varphi(y, \bar{v}_0 - Gy)$. For the golden section search we iterate from an initial interval $[\bar{v}_1, \bar{v}_3]$ by setting $\bar{v}_2 = \bar{v}_1 + (\bar{v}_3 - \bar{v}_1)(3 - \sqrt{5})/2$ and evaluating $F_1 = F(\bar{v}_1)$, $F_2 = F(\bar{v}_2)$ and $F_3 = F(\bar{v}_3)$ to obtain an initial triplet $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and the corresponding objective function values (F_1, F_2, F_3). Then we proceed with the standard golden search to iteratively obtain new triplets $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ until $\bar{v}_3 - \bar{v}_1 < \texttt{tol}$. Correspondingly, for the Newton iteration with an initial guess $\bar{v}_{\text{old}}$ we evaluate F at the three points $(\bar{v}_{\text{old}} - h, \bar{v}_{\text{old}}, \bar{v}_{\text{old}} + h)$ (where h is a small finite-difference deviation) and choose as the new point $\bar{v}_{\text{new}}$ the maximum point of the interpolating parabola. The golden section search is known to converge globally for unimodal functions with rate $(\sqrt{5}-1)/2$, whereas the Newton iteration converges quadratically close to a local maximum. In the presence of random disturbances in the evaluation of F we gradually increase the accuracy of the evaluation of F during the iteration by using the mean of y over a longer time interval after the transients have settled. Note that, when using act-and-wait and a discrete-time optimization algorithm one has removed two of the two slower time scales present in [4], replacing them with the algorithmic iteration (the Newton iteration converges super-exponentially).

5 Extremum seeking for the chemostat

Reference [42] has shown that there exists an output feedback that satisfies the assumptions A1, A2 and A4 for the model (1), with a simple saturated proportionate controller. Define first the saturation function as follows.

$$\text{sat}_{[D_{\min}, D_{\max}]}(\xi) = \left| \begin{array}{ll} D_{\max} & \text{if } \xi > D_{\max}, \\ \xi & \text{if } \xi \in [D_{\min}, D_{\max}], \\ D_{\min} & \text{if } \xi < D_{\min}, \end{array} \right. \tag{23}$$

For this saturation function [42] proved the following proposition.

Proposition 1 *Assume that the reference values* $(\bar{s}, \bar{D})$ *are in* $[s_{\min}, s_{\text{in}}) \times [D_{\min}, D_{\max}]$, *where the bounds satisfy*

$$\begin{array}{ll} D_{\min} & < \mu(s) \quad \text{for all } s \in [s_{\min}, s_{\text{in}}], \\ D_{\max} & > \mu(s) \quad \text{for all } s \in [0, s_{\text{in}}]. \end{array} \tag{24}$$

Then the output feedback

$$u(y, \bar{D}, \bar{s}) = sat_{[D_{\min}, D_{\max}]} \left(\bar{D} - G_1(y - \bar{s}) \right) \tag{25}$$

with a gain G_1 satisfying

$$G_1 > \max\left(\max_{s \in [0, s_{\text{in}}]} -\mu'(s), \; \frac{\bar{D} - \mu(s_{\text{in}})}{s_{\text{in}} - \bar{s}}\right) \tag{26}$$

guarantees that the closed-loop dynamics (1), (25) has a stable equilibrium $(s_{\text{eq}}, b_{\text{eq}}) \in [0, s_{\text{in}}) \times (0, \infty)$, which attracts all initial conditions $(s(0), b(0)) \in [0, s_{\text{in}}) \times (0, \infty)$.

See [42] for a proof.

From system (1), one has at steady state

$$\mu(s_{\text{eq}}) = \bar{D} - G_1(s_{\text{eq}} - \bar{s}), \tag{27}$$

and consequently one has

$$s_{\text{eq}} = \bar{s} \; \Leftrightarrow \; \bar{D} = \mu(s_{\text{eq}}), \tag{28}$$

that is, Assumption A2 is fulfilled.

Note that in system (1) at steady state one has $b = s_{\text{in}} - s$. Thus, the objective function

$$\phi(s) = \mu(s)(s_{\text{in}} - s), \tag{29}$$

which is non-negative such that $\phi(0) = \phi(s_{in}) = 0$, is equivalent to the production rate $\mu(s)b$. For ϕ the assumptions A3 and A4 are fulfilled.

5.1 Implementation of continuous-time adaptation

For the continuous-time adaptation we can choose the same adaptive scheme for $\bar{u}$ for the step 1 described in Section 3.1 as was chosen in [42] for the identification of μ.

Proposition 2 *For any $\bar{s} \in (0, s_{in})$ and numbers $D_{\min}$, $D_{\max}$ such that $0 < D_{\min} < \mu(\bar{s}) < D_{\max}$, the dynamical feedback law*

$$\begin{aligned} u(y, \bar{D}, \bar{s}) &= sat_{[D_{\min}, D_{\max}]}\left(\bar{D} - G_1(y - \bar{s})\right) \\ \dot{\bar{D}} &= -G_2(y - \bar{s})(\bar{D} - D_{\min})(D_{\max} - \bar{D}) \end{aligned} \tag{30}$$

exponentially stabilizes the system (1) locally about $(\bar{s}, s_{in} - \bar{s})$, for any positive constants (G_1, G_2) such that $G_1 > -\mu'(\bar{s})$. Furthermore one has

$$\lim_{t \to +\infty} \bar{D}(t) = \mu(\bar{s})$$

See [42] for the proof.

Our continuous-time extremum seeking output feedback for the chemostat model (1) is thus given by the following equations

$$u(y, \bar{D}, \bar{s}) = sat_{[D_{\min}, D_{\max}]}\left(\bar{D} - G_1(y - \bar{s})\right) \tag{31}$$

$$\dot{\bar{D}} = -G_2(y - \bar{s})(\bar{D} - D_{\min})(D_{\max} - \bar{D}) \tag{32}$$

$$\frac{1}{\epsilon}\dot{\bar{s}} = \text{sgn}\left[(\bar{D}(s_{in} - \bar{s}) - \bar{D}(t - \tau)(s_{in} - \bar{s}(t - \tau)))(\bar{s} - \bar{s}(t - \tau))\right] \tag{33}$$

5.2 Implementation of discrete-time optimization

The discrete-time optimization uses only the saturated feedback, combining the parameters $\bar{D}$ and $\bar{S}$ into a single parameter $\bar{v} = \bar{D} + G_1 \bar{s}$, such that

$$u(y, \bar{v}) = \text{sat}_{[D_{\min}, D_{\max}]}(\bar{v} - G_1 y). \tag{34}$$

If the initial bracketing interval $[\bar{v}_1, \bar{v}_2]$ for $\bar{v}$ is chosen such the line $\ell(\bar{v}) = \{(\bar{D}, \bar{s}) : \bar{D} + G_1 \bar{s} = \bar{v}\}$ intersects the domain of admissible reference values $[s_{\min}, s_{\text{in}}) \times [D_{\min}, D_{\max}]$ for all $\bar{v} \in [\bar{v}_1, \bar{v}_2]$, Proposition 1 ensures the existence of a unique equilibrium $s_{\text{eq}}(\bar{v})$ for all $\bar{v} \in [\bar{v}_1, \bar{v}_2]$. Thus, we can proceed with the golden section search for the objective functional

$$F(\bar{v}) = u(s_{\text{eq}}(\bar{v}), \bar{v})[s_{\text{in}} - s_{\text{eq}}(\bar{v})], \tag{35}$$

where u, as defined in (34), is the asymptotic value of the input u. Once, the golden section search has shrunk the bracketing interval to a certain size, we apply a single step of the Newton iteration for the objective functional F.

6 Numerical simulations

The output feedback law (31)–(33) has been simulated for a Haldane function

$$\mu(s) = \frac{\mu_{\max} s}{K + s + s^2/K_i} \tag{36}$$

and the following values of the parameters:

$\mu_{\max}$	K	K_i	S_{in}	$D_{\min}$	$D_{\max}$
1.0	1.0	0.1	1.0	0.0	1.0

Note however, that in our simulations we make no assumptions about the type of μ apart from those required in the assumptions of propositions 1 and 2 (leading to the validity of assumptions A1–A4). For the control laws in (31)–(33), the following parameters have been chosen:

G_1	G_2	ϵ
1.0	1.0	$1e^{-3}$

To simulate measurement noise, we have considered that the output is corrupted in the following way

$$y(t) = s(t)(1 + b(t)) \tag{37}$$

where $b(\cdot)$ is a random square signal of frequency ω and amplitude a, whose values are given below.

ω	a
0.2	0.05

6.1 Continuous-time adaptation

Figure 1 shows how the trajectory of the variables $(\bar{s}, \bar{D})$ converges to the graph of the function $\mu(\cdot)$ i nthe $(\bar{s}, \bar{D})$-plane initially up to disturbances due to the measurement noise. This convergence toward $\mu(\cdot)$ is effected by the control laws (31) and (32) on the fast timescale (for $t \ll 100$, not visible in the time series depicted in Figure 2). Once the feedback laws (31) and (32) have achieved quasi-equilibrium for fixed $\bar{s}$ the dynamics of (33) maximizes the functional ϕ on the slow timescale. Timescale separation $\epsilon = 10^{-3}$ is chosen for our simulations. In Figures 1 and 3 one can also see that the adaptation of $\bar{s}$ via (33) forces the state (s, b) to converge near the equilibrium $(s^\star, b^\star)$ (indicated as intersection between dotted line and graph $\mu(\cdot)$, where $s^\star$ is maximizing the function $\phi(\cdot)$. We note that the adaptation has converged already after $t = 3500$.

   Remark If we choose the timescales not sufficiently different (say, $\epsilon = 10^{-2}$ in our example) we observe that the chemostat (1) with feedback laws (31)–(33) becomes dynamically unstable in in its fixed point $(s, b, \bar{s}, \bar{D}) = (s^\star, b^\star, s^\star, \mu(s^\star))$.
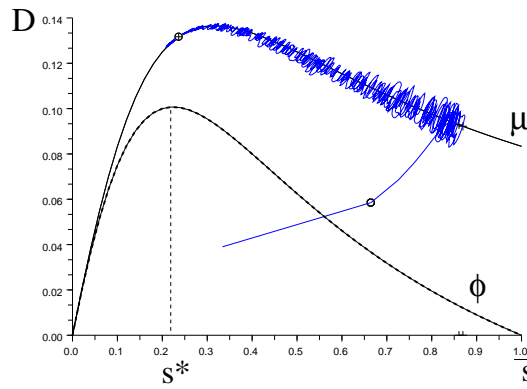
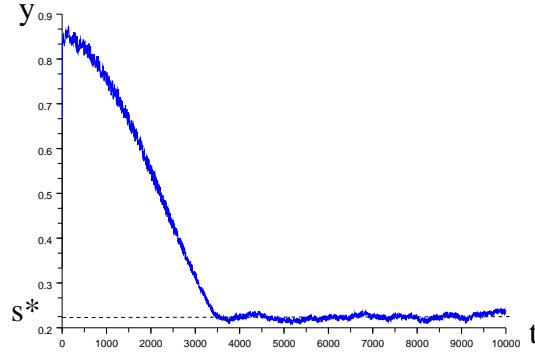Figure 1: Trajectory of (1) with feedback laws (31)–(33) in the $(\bar{s}, \bar{D})$ plane.

Figure 2: Output of (1) with feedback laws (31)–(33), disturbed by (37), over time (same run as Figure 1).

6.2 Discrete-time algorithms

For the discrete-time optimization algorithm we chose an initial bracketing interval $[\bar{v}_1, \bar{v}_2] = [0.04, 1]$ (gain $G_1 = 1$ in feedback law (34) identical to the continuous-time adaptation). During the golden section search (until $t \approx 700$ in Figure 4(a)) we checked every $t_{\text{inc}} = 25$ time units if the transients have settled (criterion is a decrease of the standard deviation compared to the previous time interval). Once the output is accepted as stationary we used the mean of output and input over the last time interval to calculate the objective F. A larger t_{inc} inceases the accuracy of the evaluation of F in the presence of disturbances due to averaging. We observe that after 7 evaluations of F the golden section search reduces the bracketing interval to length 0.2 and the estimated optimal value of s is already close to $s^\star$ ($t \approx 700$ in Figure 4(a)). The Newton step then only confirms the optimality of the final estimate of $s^\star$ (up to disturbance level). For the Newton iteration we chose t_{inc} larger ($t_{\text{inc}} = 100$) to increase the accuracy of F evaluations and use the points $(\bar{v} - h, \bar{v}, \bar{v} + h)$ with $h = 0.05$ to evaluate the approximating parabola for F in $\bar{v}$. Figure 4(b) shows the trace (grey dots) of outputs s and inputs u (equaling dilution rate D) in the (s, D)-plane. The gaps in the trace are due to finite sampling time. The full circles show the points where the transients were accepted as settled.

The speed and accuracy of the discrete-time optimization is limited by the choice of t_{inc}, which in turn has to be determined sufficiently large to enable an averaging effect for the disturbances.
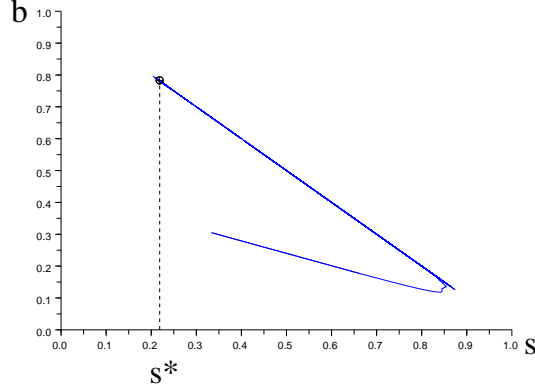
8

Figure 3: Trajectory of same run as Figures 1 and 2 in the (s, b) plane ($\{(s, b) : s + b = s_{\text{in}}\}$ is invariant).

7 Conclusion

We have proposed two extremum-seeking schemes: one approach uses dynamical output feedback that is based on a continuation method. Its closed-loop dynamics possesses two time scales: the timescale of the original system with stabilizing feedback and a slow timescale for the adaptation scheme. As an alternative we test classical optimization algorithms, applied to the steady-state outputs of the feedback-stabilized system, which have a continuous timescale (which is exponentially convergent) and a discrete timescale (which is exponentially or superexponentially convergent). This is in contrast to extremum seeking techniques that use dither signals, which need three timescales for nonlinear systems (the time scale of the stabilized system is the fastest). We illustrate our approaches on the chemostat model for unknown growth function and single measurement. The numerical simulations suggest that the method is practically viable and robust with respect to measurement disturbances. Extensions and a detailed convergence analysis for general systems and the continuous-time adaptation will be the subject of future work. Conditions for the convergence of the discrete-time optimization can be deduced directly from the corresponding statements for the original numerical algorithms.

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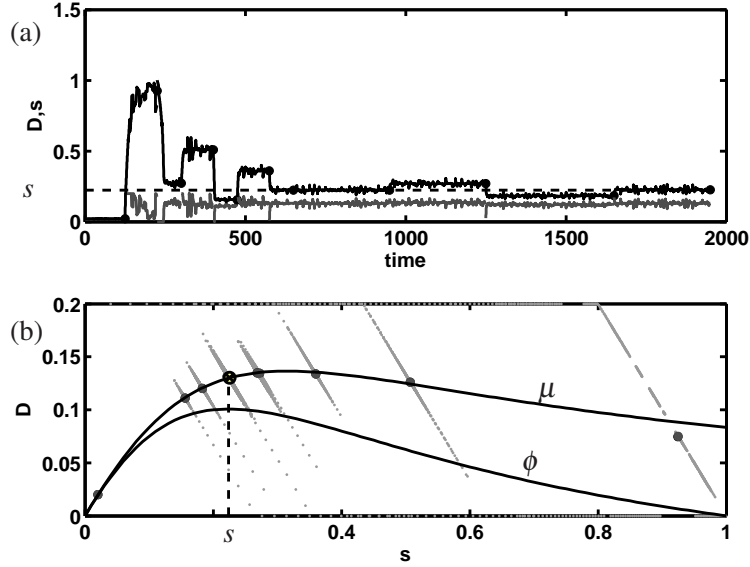
Figure 4: Time profiles (a) and input-output plane (b) for time-discrete optimization. The grey time profile in (a) is $u(t) = D(t)$ and the black profile is $s(t)$. Full circles in (a) and (b) indicate inputs and outputs where the objective F was evaluated.

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